Stirling probability and ***q***-bosons

1998 J. Phys. A: Math. Gen. 31 1121

(http://iopscience.iop.org/0305-4470/31/4/003)

J. Phys. A: Math. Gen. **31** (1998) 1121–1125. Printed in the UK PII: S0305-4470(98)85717-X

Stirling probability and q-bosons

M Arik and G Ünel

Boğaziçi University, Physics Department Bebek 80815, Istanbul, Turkey

Received 7 July 1997, in final form 24 September 1997

Abstract. Normal ordering of powers of the bosonic number operator can be used to define a discrete probability distribution associated with the number of elements of a random set. We represent these random sets by vectors in a Hilbert space and obtain q-bosons.

The connection between the Stirling numbers [1] and the normal ordering of the quantum operators has been widely investigated. Recently, q deformation of the same problem in connection with quantum groups has attracted much attention. Derivation of the q deformed versions of combinatorial identities [2] and of q deformed bosons [3] are extensively studied. Stirling numbers are defined through the expansion formula:

$$M^n = \sum_{m=0}^{n} S_m^n M(M-1)\cdots(M-m+1). \tag{1}$$

In this expansion the coefficients S_m^n are called the Stirling numbers of the second kind. They have a recurrence relation given by:

$$S_m^n = m\, S_m^{n-1} + S_{m-1}^{n-1}. \tag{2}$$

These coefficients have the properties

$$\begin{aligned} &S_m^n \geqslant 0 \\ &S_m^n = 0 \qquad \text{iff } n < m \\ &S_0^n = \delta_0^n. \end{aligned} \tag{3}$$

These properties together with (1) motivate us to define a probability function which we will call the Stirling probability distribution.

$$P_m^n = S_m^n \frac{M!}{(M-m)!} \frac{1}{M^n}. \tag{4}$$

The normalization condition ($\sum_m P_m^n = 1$) of the Stirling probability is identically satisfied using the defining equation (1) of Stirling numbers. The probability distribution defined in (4), through purely algebraic steps, turns out to be useful in physics when applied to some interesting problems, such as q deformations and normal ordering.

To see the close connection with other subjects, let us define the same probability using the random set concept. Random sets are only distinguished by the property of having been built from another set or another random set in a given number of steps. We start with an infinite set Ω of bosons. All the bosons are considered to be identical but they can be counted in accordance with the usual physical principles. This implies that these

0305-4470/98/041121+05$19.50

identical bosons must have another attribute, that we call metacolour, which is assumed to be different for all bosons in the infinite source set Ω. We consider a set A made from Ω in N steps, by adding at each step a copy of a random element of Ω to A. The set A will have exactly $M = N$ bosons since the probability of choosing the same element twice from an infinite set is zero. Hence the set A becomes a finite set of N elements which we identify with a set of N identical bosons.

In quantum mechanics, this set is described by a vector $|N\rangle$ and associated with it are the annihilation, creation and number operators a, $a^\dagger$ and $\hat{N}$ which satisfy:

$$\begin{aligned} \hat{N}|N\rangle &= N|N\rangle \\ a^\dagger|N\rangle &= \alpha_N|N+1\rangle \\ \hat{N} &= a^\dagger a. \end{aligned} \tag{5}$$

We identify $a^\dagger$ with the operation of choosing an element from the source set Ω and putting a copy of it into the set A. It can be shown that operators in (5) satisfy:

$$\begin{aligned} aa^\dagger - a^\dagger a &= 1 \\ a\hat{N} &= (\hat{N}+1)a. \end{aligned} \tag{6}$$

We now consider another set B made from the finite set A, in n steps. (Note that the elements in A are all identical bosons except their metacolour which counts the cardinality $M = N$ of this set.) In each step, a copy of an element randomly chosen from A is put into B if it does not already exist in B. If a boson of the same metacolour already exists in B, the number of elements in B does not increase. Mathematically speaking this amounts to choosing a random subset of A of cardinality one and taking its union with B. Starting from a finite set of M elements the probability of having m elements in B after n steps will be denoted by $P^M_{n,m}$. Using a simple reasoning, this probability can be defined in the following recursive way

$$P^{M,n}_m = \left(\frac{m}{M}\right) P^{M,m-1}_m + \left(1 - \frac{m-1}{M}\right) P^{M,n-1}_{m-1}. \tag{7}$$

This indeed yields the Stirling probability in (4). A schematic view of the above discussion is given in figure 1. As $a^\dagger$ was the creation operator for the set A, we define $b^\dagger$ as the same for the random set B. The number operator $a^\dagger a$ gives the number of elements in A, thus

$$\hat{M} = a^\dagger a. \tag{8}$$

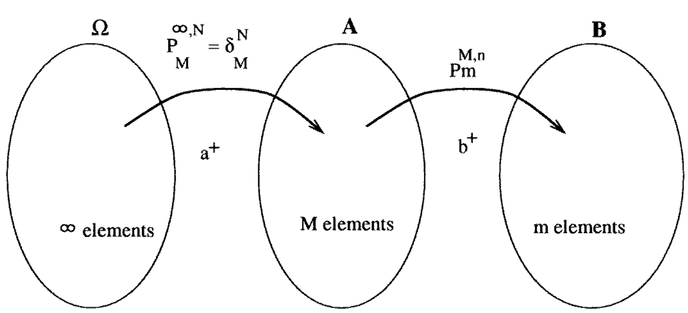

Figure 1. The infinite source set Ω, the finite set A and the random set B.

Since the source set of A has infinite number of elements, the commutation relation between a and $a^\dagger$ can be written as:

$$aa^\dagger = a^\dagger a + 1. \tag{9}$$

The average number of elements in B after n steps can also be easily computed:

$$\langle m \rangle \equiv \sum_{m=0}^{n} m P_m^n = \frac{1-q^n}{1-q} = [n]_q \tag{10}$$

which turns out to be a Jackson [4] basic number, where

$$q \equiv 1 - \frac{1}{M} \qquad \hat{q} = 1 - (a^\dagger a)^{-1}. \tag{11}$$

$\hat{q}$ is defined only if the set A is not null.

Before proceeding to investigate the properties of the operator algebra of a and b , let us see the connection of this problem with normal ordering of quantum operators. We start with an oscillator algebra with lowering and raising operators a and $a^\dagger$. If we want to express the nth power of the number operator $a^\dagger a$ as a normal ordered expansion, we can write:

$$(a^\dagger a)^n = \sum_{m=0}^{n} S_m^n (a^\dagger)^m (a)^m. \tag{12}$$

Multiplying this equation by $a^\dagger a$ from the right-hand side and using the commutation relation between the creation and annihilation operators, we obtain a recurrence relation for S_m^n, which turns out to be (2). Formula (1) becomes the matrix element of (12) obtained by taking the expectation value in the state $|M\rangle$.

The parameter q defined in (12) appears naturally in the commutation relation between b and $b^\dagger$ [5]:

$$bb^\dagger = qb^\dagger b + 1. \tag{13}$$

This is the q-oscillator [10] algebra which leads to the quantum group [11] $U_q(d)$ by considering d random sets made from A whose union gives B [5]. Then the number operator for the quantized random set B becomes

$$\hat{m} = b^\dagger b \tag{14}$$

whose eigenvalues are the average number of elements in B.

After setting a proper algorithm to 'quantize' the finite sets, we can go one step further and 'second quantize' the random set B to obtain a new set that we will call C. This new

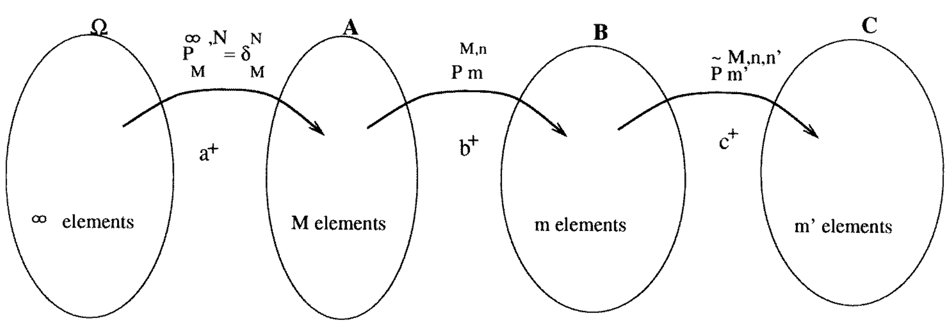

Figure 2. Second quantization of the finite set A.

set will be formed from B using the probability function $\tilde{P}$ and we will have m' elements after n' steps. This is illustrated in figure 2. Unfortunately, the algorithm to define $\tilde{P}$ is not unique. One method is to define this probability through the q-oscillator algebra's commutation relation of the source set of C, since the Stirling probability $P^M_{n,m}$, could also be defined using (9) and (7). Thus we can write

$$(b^\dagger b)^n = \sum_{m=0}^{n} \tilde{S}^n_m (b^\dagger)^m (b)^m \tag{15}$$

to define q Stirling numbers of the second kind. These numbers were first introduced in 1933 [6] and extensively studied by various authors [7–9]. We will use (13) to find the recurrence relation concerning $\tilde{S}$ as

$$\tilde{S}_{n,m} = q^{m-1}\tilde{S}_{n-1,m-1} + [m]\tilde{S}_{n-1,m} \tag{16}$$

which permits us to define $\tilde{P}$ as:

$$\tilde{P}^{n,m}_{n',m'} = \tilde{S}^{n'}_{m'} \frac{[n]!}{[n-m']![n]^{n'}}. \tag{17}$$

Using this probability distribution we can calculate the average number of elements in C.

An alternative method to this approach is to define the probability of having m' elements in the set C similar to the probability of having m elements in the set B. Then the final probability can be written as a summation containing these two probabilities:

$$\tilde{P}^{M,n,n'}_{m'} \equiv \sum_m P^{M,n}_m P^{m,n'}_{m'}. \tag{18}$$

This is the classical probability of having m' elements in set C which is obtained by starting with set A of M elements and forming set B in n steps by choosing elements from A and then forming set C in n' steps by choosing elements from B.

This new definition gives us a new mean value for the number of elements in C. If we first evaluate the summation over m' we obtain:

$$\langle m' \rangle \equiv \sum_{m'} m' \tilde{P}^{M,n,n'}_{m'} = \sum_m P^{M,n}_m \frac{1-q(m)^{n'}}{1-q(m)} \tag{19}$$

which is clearly the average of the basic number n' with parameter q as a function of m. If we approximate the average of a function by the function of the average we obtain for the average number of elements in C

$$\langle m' \rangle = \left\langle \frac{1-q(m)^{n'}}{1-q(m)} \right\rangle \approx \frac{1-q(\langle m \rangle)^{n'}}{1-q(\langle m \rangle)} = \frac{1-\tilde{q}^{n'}}{1-\tilde{q}} \tag{20}$$

where

$$\tilde{q} \equiv q(\langle m \rangle) = 1 - \frac{1}{\langle m \rangle} = 1 - \frac{1}{[n]_q}. \tag{21}$$

This last expression for the average number of elements in C gives different results compared with the values calculated using (17). But note that both expressions converge to the same values in the $q = 1$ ($N = \infty$) limit. The advantage of (20) and (21) is that second generation random sets may again be described by q-oscillators which generate Jackson basic numbers for the average number of elements.

In this work we have discussed the relationship between the q-oscillators [10] and $U_q(d)$ quantum groups [11] using the random set concept. We have seen that association of vectors in a Hilbert space with the random sets, yields a new property of q-bosons, that we call metacolour. Metacolour permits us to count the identical q-bosons and other bosons which correspond to $q = 1$. On the other hand, for the random or quantized random sets, state vectors are defined by the number of steps used to construct this set rather than the uncertain number of elements in this set. Furthermore, the vectors corresponding to sets constructed in different number of steps are orthogonal. This probabilistic view is different from the classical definition of a state according to the final number of particles in that state. Since the concept of quantizing the random sets is rather general, we cannot define a unique procedure for second and further generation of random sets. However, note that the procedure defined by (19) and (20) gives a simple and unique answer. We believe that the properties of q-bosons related to the metacolour and probability concepts are interesting and worth deeper investigation.

References

[1] Speigel M R 1971 *Calculus of Finite Differences and Difference Equations* (New York: McGraw-Hill)
[2] Katriel J and Duchamp G 1995 *J. Phys. A: Math. Gen.* **28** 7209
[3] Katriel J and Kibler M 1992 *J. Phys. A: Math. Gen.* **25** 2683
[4] Jackson F H 1951 *Q. J. Math. Oxford Ser.* 2
[5] Arik M, Kornfilt J and Yildiz A 1997 *Phys. Lett.* A to appear
[6] Carlitz L 1933 *Trans. Am. Math. Soc.* **35** 122
[7] Gloud H W 1961 *Duke Math. J.* **28** 281
[8] Garvan F and Stanton D 1990 *Math. Comput.* **55** 299–311
[9] Sagan E B 1992 *Adv. Math.* **95** 127–43
[10] Arik M and Coon D D 1976 *J. Math. Phys.* **17** 524
[11] Faddeev L D, Reshetikin N Y and Tahtajan L A 1989 *Algebr. Anal.* **1** 178
Woronowicz S L 1987 *Commun. Math. Phys.* **111** 613